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by S. I. Osipov

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EXPANSION OF POLYNOMIALS IN BESSEL OPERATORS

S. I. Osipov

Let $D = d/dt$ be a differential operator, $B_k = ((1/t)\ (tD)^k$ a /149
Bessel operator (see [1]), $k = 2, 3, \ldots$, and let $P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ be a polynomial with arbitrary constant coefficients.

In this paper we will expand the operator $P_n(B_k)$ in powers of D* and we apply this expansion to investigation of ordinary linear differential equations.

Let G_n be a set of functions of t that have n-th order derivatives, $n = 1, 2, \ldots$, and assume that /150

$$\Phi\ (n, k, D) = \sum_{\nu=1}^{nk} Q\ (n, k, \nu, t)\ D^{\nu} + a_n,$$

$$Q\ (n, k, \nu, t) = \sum_{p=E((\nu+k-1)/k)}^{\min(\nu,\ n)} C\ (k, \nu, p)\ a_{n-p}\ t^{\nu-p},$$

*The problem of expanding operators coinciding with Bessel operators or operators with similar structure has been discussed in a number of papers by other authors. See, for example, 1) Issledovaniya po integro-differentsial'nym uravneniyam v Kirgizii (Investigations of integro-differential equations in Kirgiz), Vol. II, Frunze, 1962, pp. 300, 310-311; 2) Materialy sed'moy nauchnoy konferentsii kafedry vysshey matematiki Frunzenskogo politekhnicheskogo instituta (Proceedings of the Seventh Scientific Conference of the Departments of Higher Mathematics of Frunze Polytechnic Institute, 1963, pp. 87-90; 3) J. Math. Anal. and Appl., Vol. 6, No. 3, 1963, pp. 395-397; 4) J. Riordan. Introduction to Combinatorial Analysis [Russian translation], For. Lit. Publ. House, Moscow, p. 57, problem 18.

$$\left.\begin{aligned} C(k, \nu, p) &= \frac{1}{[(\nu - p)!]^2} \sum_{m=\max(\nu-p,p)}^{p(k-1)} \frac{m!S(k, p, m)}{(m + p - \nu)!}, \\ S(k, p, m) &= (-1)^{pk} \sum_{r=p}^{m} \frac{(-1)^{r-p}\, m!\, (r!)^{k-2}}{[(r - p)!]^{k-1}\,(m - r)!}, \end{aligned}\right\} \quad (1')$$

$$E\left(\frac{\nu + k - 1}{k}\right) \text{ - integral part of } \frac{\nu + k - 1}{k}, \; a_0 = 1. \qquad (1)$$

We have the following

Theorem. On the set G_{nk} the operators $P_n(B_k)$ and $\Phi(n, k, D)$ are equivalent, i.e., for any function $g(t) \in G_{nk}$,

$$P_n(B_k)\, g(t) = \Phi(n, k, D)\, g(t). \qquad (2)$$

Proof. It is not difficult to use induction to prove that the following equation is true for the set G_{pk}:

$$B_k^p = \sum_{\nu=p}^{pk} A(k, \nu, p)\, t^{\nu-p} D^{\nu}, \qquad (3)$$

where $A(k, \nu, p)$ are coefficients that are independent of t.

In order to determine these coefficients, we introduce the following polynomial of degree $p(k - 1)$:

$$W_{p,k}(t) = e^t B_k^p e^{-t}. \qquad (4)$$

Let

$$L_m(t) = \sum_{r=0}^{m} \frac{(-1)^r m!}{(r!)^2 (m-r)!} t^r$$

be an m-th order Laguerre polynomial (see [2], p. 110).

It is clear that

$$\int_0^\infty e^{-t} L_m(t) W_{p,k}(t)\, dt = 0 \qquad (5)$$

when $m > p(k-1)$.

It is not difficult to obtain

$$\int_0^\infty e^{-t} L_m(t) W_{p,k}(t)\, dt = (-1)^{pk} \int_0^\infty e^{-t} B_k^p L_m(t)\, dt$$

by integrating by parts and, consequently,

$$\int_0^\infty e^{-t} L_m(t) W_{p,k}(t)\, dt = \begin{cases} 0, & \text{if } m < p, \\ (-1)^p S(k, p, m), & \text{if } p \le m. \end{cases} \qquad (6)$$

By virtue of (5) and (6),

$$W_{p,k}(t) = \sum_{m=p}^{p(k-1)} (-1)^p S(k, p, m) L_m(t),$$

i.e., /151

$$W_{p,k}(t) = \sum_{m=p}^{p(k-1)} \sum_{r=0}^{m} \frac{(-1)^{r+p} S(k, p, m)\, m!}{(r!)^2 (m-r)!} t^r. \qquad (7)$$

The function

$$\delta\ (m, p) = \begin{cases} 0, & \text{if } m < p, \\ 1, & \text{if } m \geq p, \end{cases} \tag{8}$$

can be used to represent Eq. (7) in the form

$$W_{p,k}(t) = \sum_{m=0}^{p(k-1)} \sum_{r=0}^{m} \frac{(-1)^{r+p}\ \delta\ (m, p)\ S\ (k, p, m)\ m!}{(r!)^2\ (m-r)!}\ t^r,$$

and a change in the order of summation yields

$$W_{p,k}(t) = \sum_{r=0}^{p(k-1)} \sum_{m=r}^{p(k-1)} \frac{(-1)^{r+p}\ \delta\ (m, p)\ S\ (k, p, m)\ m!}{(r!)^2\ (m-r)!}\ t^r.$$

The substitution $r = \nu - p$ leads to the equation

$$W_{p,k}(t) = \sum_{\nu=p}^{pk} \sum_{m=\nu-p}^{p(k-1)} \frac{(-1)^{\nu}\ \delta\ (m, p)\ S\ (k, p, m)\ m!}{[(\nu-p)!]^2\ (m+p-\nu)!}\ t^{\nu-p}. \tag{9}$$

It follows from (3), (4), and (9) that

$$A(k, \nu, p) = \sum_{m=\nu-p}^{p(k-1)} \frac{\delta\ (m, p)\ S\ (k, p, m)\ m!}{[(\nu-p)!]^2\ (m+p-\nu)!},$$

i.e.,

$$A(k, \nu, p) = C(k, \nu, p). \tag{10}$$

In order to complete the proof of the theorem, note that by virtue of (3) and (10), we have

$$P_n(B_k) - a_n = \sum_{p=1}^{n} \sum_{\nu=p}^{pk} C(k, \nu, p)\ a_{n-p} t^{\nu-p} D^{\nu}.$$

The function (8) can be used to represent this last expression in the form

$$P_n(B_k) - a_n = \sum_{p=1}^{nk} \sum_{\nu=p}^{nk} \delta\ (n, p)\ \delta\ (pk, \nu)\ C\ (k, \nu, p)\ a_{n-p} t^{\nu-p} D^{\nu},$$

and, changing the order of summation, we obtain

$$P_n(B_k) - a_n = \sum_{\nu=1}^{nk} \sum_{p=1}^{\nu} \delta\ (n, p)\ \delta\ (pk, \nu)\ C\ (k, \nu, p)\ a_{n-p} t^{\nu-p} D^{\nu}.$$

It only remains to note that

$$\sum_{p=1}^{\nu} \delta\ (n, p)\ \delta\ (pk, \nu)\ C\ (k, \nu, p)\ a_{n-p} t^{\nu-p} = Q\ (n, k, \nu, t),$$

which completes the proof.

We shall consider the special case $k = 2$ in more detail. Here we are dealing with the operator $(d/dt)\ t\ (d/dt)$, and the operator $\Phi\ (n, 2, D)$ can be used to expand a polynomial in this operator.

Noting that

$$C\ (2, \nu, p) = \frac{p!\ S\ (2, p, p)}{[(\nu - p)!]^2\ (2p - \nu)!}, \quad S\ (2, p, p) = p!,$$

we find that /152

$$C\ (2,\ \nu,\ p) = \frac{(p!)^2}{[(\nu - p)!]^2\ (2p - \nu)!}, \tag{11}$$

and

$$\Phi\ (n,\ 2,\ D) = \sum_{\nu=1}^{2n} Q\ (n,\ 2,\ \nu,\ t)\ D^\nu + a_n,$$

so that

$$Q\ (n,\ 2,\ \nu,\ t) = \sum_{p=E((\nu+1)/2)}^{\min(\nu,n)} \frac{(p!)^2\ a_{n-p}}{[(\nu - p)!]^2\ (2p - \nu)!}\ t^{\nu-p}.$$

This result also follows from the theorem discussed in [3]* when we apply it to the operator $(d/dt)\ t\ (d/dt)$.

Consider the coefficients $C\ (k,\ \nu,\ p)$, defined by formulas (1'). When $k = 2$ these coefficients, as (11) shows, have a simple structure, and it is not difficult to evaluate them. In the general case ($k = 3, 4, \ldots$) formulas (1') are rather complex. As a result, it may prove useful to use the following algorithm for evaluation of the coefficients $C\ (k,\ \nu,\ p)$:

$$C\ (k,\ p,\ p) = (p!)^{k-1},\quad C\ (k,\ p + 1,\ p) = [(p + 1)!]^{k-1} - C\ (k,\ p,\ p),$$

and, more generally, for $\nu = p + 1,\ p + 2,\ \ldots,\ pk$,

*It should be borne in mind that the notation used in this paper is different from that used in [3]. The author of [3] used B_α to denote the operator $t^{-\alpha}\ Dt^{1+\alpha}\ D$.

$$C(k, \nu, p) = \frac{(\nu!)^{k-1}}{[(\nu - p)!]^k} - \frac{1}{1!} C(k, \nu - 1, p) - \frac{1}{2!} C(k, \nu - 2, p) - \cdots \tag{12}$$

$$\cdots - \frac{1}{(\nu - p)!} C(k, p, p)$$

for $k = 2, 3, \ldots$ and $p = 1, 2, \ldots$.

We shall now prove (12). By (10) and (3), the coefficients $c(k, \nu, p)$ are numbers such that for any function $g(t) \in G_{pk}$

$$B_k^p \, g(t) = \sum_{\nu=p}^{pk} C(k, \nu, p) \, t^{\nu-p} D^\nu g(t). \tag{13}$$

We take the function $I^{(k-1)}_{0,0,\ldots,0}(k\sqrt[k]{t})$ from [1] as $g(t)$. Since it satisfies the differential equation

$$\frac{1}{t} \left(t \frac{d}{dt}\right)^k y = y,$$

we have

$$B_k^p \, I^{(k-1)}_{0,0,\ldots,0}(k\sqrt[k]{t}) = I^{(k-1)}_{0,0,\ldots,0}(k\sqrt[k]{t}), \tag{14}$$

so that the left side of this equation can be treated as the result of p-fold application of the operator $(d/dt)(t(d/dt))^{k-1}$ to the function $I^{(k-1)}_{0,0,\ldots,0}(k\sqrt[k]{t})$. Noting that

$$I^{(k-1)}_{0,0,\ldots,0}\ (k\sqrt[k]{t}) = \sum_{r=0}^{\infty} \frac{t^r}{(r!)^k},$$

we can derive the following expression from (13) and (14):

$$\sum_{\nu=p}^{pk} C(k,\ \nu,\ p)\ t^{\nu-p}\ D^{\nu} \sum_{r=0}^{\infty} \frac{t^r}{(r!)^k} = \sum_{r=0}^{\infty} \frac{t^r}{(r!)^k}. \tag{15}$$

But /153

$$\sum_{\nu=p}^{pk} C(k,\ \nu,\ p)\ t^{\nu-p}\ D^{\nu} \sum_{r=0}^{\infty} \frac{t^r}{(r!)^k} = \sum_{r=0}^{\infty} \left\{\sum_{\nu=p}^{pk} C(k,\ \nu,\ p)\ t^{\nu-p}\ D^{\nu}\right\} \frac{t^r}{(r!)^k} =$$

$$= \sum_{r=p}^{\infty} \left\{ \sum_{\nu=p}^{\min(r,\ pk)} C(k,\ \nu,\ p)\ \frac{1}{(r-\nu)!}\right\} \frac{t^{r-p}}{(r!)^{k-1}} =$$

$$= \sum_{r=0}^{\infty} \left\{ \sum_{\nu=p}^{\min(r+p,\ pk)} C(k,\ \nu,\ p)\ \frac{1}{(r+p-\nu)!}\right\} \frac{t^{r}}{[(r+p)!]^{k-1}},$$

and comparison of the coefficients of the first $p(k-1) + 1$ powers of $\underline{t}$ in the right and left sides of (15) gives us

$$\sum_{\mu=p}^{r+p} C(k,\ \mu,\ p)\ \frac{1}{(r+p-\mu)!} = \frac{[(r+p)!]^{k-1}}{(r!)^k}\ (r = 0,\ 1,\ 2,\ldots,\ pk - p).$$

Substitution of $r + p$ by ν p yields

$$\sum_{\mu=p}^{\nu} C(k, \mu, p) \frac{1}{(\nu - \mu)!} = \frac{(\nu!)^{k-1}}{[(\nu - p)!]^k} \quad (\nu = p, p + 1, \ldots, pk),$$

and, consequently, formulas (12) are proved.

The operational calculus of Bessel operators can be used (under certain conditions) to integrate differential equations whose left side is of the form $P_n(B_k)x(t)$. But, both in the theory of ordinary differential equations and in practice, the left sides of differential equations are usually given in the form of a linear differential expression, i.e., in the form

$$q_0(t) \frac{d^m}{dt^m} x(t) + q_1(t) \frac{d^{m-1}}{dt^{m-1}} x(t) + \cdots + q_{m-1}(t) \frac{d}{dt} x(t) + q_m(t) x(t), \tag{16}$$

where $q_0(t), \ldots, q_m(t)$ are coefficients and $\underline{m}$ is the order of the differential equation.

In connection with this, it is interesting to investigate the problem of which linear differential equations (16) are representable in terms of $P_n(B_k)x(t)$ and to give a method for reducing (16) to the form $P_n(B_k)x(t)$ when this is possible.

It is a direct consequence of the expansion theorem for $P_n(B_k)$ that representability of (16) in the form $P_n(B_k)x(t)$ requires that $q_0(t), q_1(t), \ldots, q_{m-1}(t)$ be polynomials and that $q_m(t)$ be constant.

Let $q_0(t), \ldots, q_{m-1}(t)$ be polynomials in $\underline{t}$ and let $q_m(t)$ be a constant, $m \geq 2$ (the cases $m = 0$ and $m = 1$ are trivial). Using the expansion of $P_n(B_k)$ and considering that the number of pairs of integers (n, k) satisfying the conditions $k_n = m$, $k \geq 2$, $n \geq 1$ for any fixed number $\underline{m}$ is finite, we can always establish whether or not (16) is representable in the form $P_n(B_k)x(t)$.

In order to do this we first find all of the admissible pairs of values of $\underline{n}$ and $\underline{k}$ for a given $\underline{m}$. Among these pairs we select a pair with the largest value of $\underline{k}$ (denote it by k_1, and denote the corresponding value of $\underline{n}$ by n_1) and we calculate $E((\nu + k_1 - 1)/k_1)$ and $\min(\nu, n_1)$ for $\nu = 1, 2, \ldots, m$.

Let $\sigma_{H,\nu}$ and $\sigma_{B,\nu}$ be the smallest and largest of the exponents contained in $q_{m-\nu}(t)$. If $q_{m-\nu}(t) \equiv 0$, we put

$$\sigma_{H,\nu} = \nu - \min(\nu, n_1), \quad \sigma_{B,\nu} = \nu - E\left(\frac{\nu + k_1 - 1}{k_1}\right).$$

We then construct Table 1 until a negative number appears in column IV or VII. If such a number appears, we conclude that (16) is not representable in the form $P_{n_1}(B_{k_1})x(t)$, and we then turn to testing the next admissible pair of values of $\underline{n}$ and $\underline{k}$.

TABLE 1

I	II	III	IV	V	VI	VII
ν	$\min(\nu,n_1)$	$\sigma_{H,\nu}$	$\min(\nu,n_1) +$ $+ \sigma_{H,\nu} - \nu$	$E((\nu+k_1-1)/k_1$	$\sigma_{B,\nu}$	$\nu-\sigma_{B,\nu}-$ $-E((\nu+k_1-1)/k_1)$
1						
.						
.						
.						
.						
m						

If all of the numbers in columns IV and VII are nonnegative, we write the polynomials $q_{m-\nu}(t)$ in the form /154

$$\sum_{p=E((\nu+k_1-1)/k_1)}^{\min(\nu,n_1)} \beta\ (\nu,\ p)\ t^{\nu-p}.$$

TABLE 2

ν \ p	1	...	n_1
1	$\beta(1,1)/C(k_1,\ 1,1)$		
.			
.			
.			
m			$\beta(m,\ n_1)/C(k_1, m, n_1)$

TABLE 3

I	II	III	IV	V	VI	VII
ν	$\min(\nu,1)$	$\sigma_{H,\nu}$	$\min(\nu,1)+\sigma_{H,\nu}-\nu$	$E((\nu+5)/6)$	$\sigma_{B,\nu}$	$\nu-\sigma_{B,\nu}-E((\nu+5)/6)$
1	1	0	0	1	0	0
2	1	0	-1			

TABLE 4

I	II	III	IV	V	VI	VII
ν	$\min(\nu,2)$	$\sigma_{H,\nu}$	$\min(\nu,2)+\sigma_{H,\nu}-\nu$	$E((\nu+2)/3)$	$\sigma_{B,\nu}$	$\nu-\sigma_{B,\nu}-E((\nu+2)/3)$
1	1	0	0	1	0	0
2	2	0	0	1	0	1
3	2	1	0	1	1	1
4	2	2	0	2	2	0
5	2	3	0	2	3	0
6	2	4	0	2	4	0

TABLE 5

ν \ p	1	2
1	0	
2	0	1
3	0	1
4		1
5		1
6		1

In this case, $\beta(\nu, p)$ is uniquely determined by the polynomial $q_{m-\nu}(t)$. We use formulas (12) to evaluate the coefficients $C(k_1, \nu, p)$ for $p = 1, 2, \ldots, n_1$ and $\nu = p, p + 1, \ldots, pk_1$. We place the fractions $\beta(\nu, p)/C(k_1, \nu, p)$ at the intersection of the ν-th row and the p-th column in Table 2.

When different fractions appear in any column we conclude that (16) is not representable in the form $P_{n_1}(B_{k_1})x(t)$ and turn to testing the next admissible pair of values for $\underline{n}$ and $\underline{k}$.

If, however, all of the fractions in each column are equal, we conclude that (16) is representable in the form $P_{n_1}(B_{k_1})x(t)$ and that the coefficient a_{n_1-p}, $p = 1, 2, \ldots, n_1$, of the polynomial $P_{n_1}(B_{k_1})$ is the number of the p-th column.

Consider, for example, the linear differential expression

$$t^4 \frac{d^6}{dt^6} x(t) + 12t^3 \frac{d^5}{dt^5} x(t) + 38t^2 \frac{d^4}{dt^4} x(t) + 32t \frac{d^3}{dt^3} x(t) + 4 \frac{d^2}{dt^2} x(t). \tag{17}$$

The admissible pairs of values of $\underline{n}$ and $\underline{k}$ are

n	1	2	3
k	6	3	2.

We first test $n_1 = 1$, $k_1 = 6$. We construct the first table for this case -- Table 3. At $\nu = 2$ we obtain a negative number in column IV: -1. As a result, (17) cannot be represented in the form $P_1(B_6)x(t)$. We now turn to testing the pair $n_2 = 2$, $k_2 = 3$.

The first table for this case -- Table 4 -- yields no negative entries and we continue the investigation. We have /155

$$q_5(t) = 0 = \sum_{p=1}^{1} \beta(1, p)\, t^{1-p}, \qquad \beta(1,1) = 0;$$

$$q_4(t) = 4 = \sum_{p=1}^{2} \beta(2, p)\, t^{2-p}, \qquad \beta(2,1) = 0, \quad \beta(2,2) = 4;$$

$$q_3(t) = 32t = \sum_{p=1}^{2} \beta(3, p)\, t^{3-p}, \quad \beta(3,1) = 0, \quad \beta(3,2) = 32;$$

$$q_2(t) = 38t^2 = \sum_{p=2}^{2} \beta(4, p)\, t^{4-p}, \qquad \beta(4,2) = 38;$$

$$q_1(t) = 12t^3 = \sum_{p=2}^{2} \beta(5, p)\, t^{5-p}, \quad \beta(5,2) = 12;$$

$$q_0(t) = t^4 = \sum_{p=2}^{2} \beta(6, p)\, t^{6-p}, \qquad \beta(6,2) = 1.$$

Computation of the coefficients $C(3, \nu, p)$ for $p = 1, 2$, and $\nu = p, \ldots$
$\ldots, 3p$ yields

$C(3, 1, 1) = 1, \quad C(3, 2, 1) = 3, \quad C(3, 3, 1) = 1, \quad C(3, 2, 2) = 4,$

$C(3, 3, 2) = 32, \quad C(3, 4, 2) = 38, \quad C(3, 5, 2) = 12, \quad C(3, 6, 2) = 1,$

and we obtain Table 5, which shows that (17) is representable in the form $(B_3)^2 x(t)$.

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